

1 Introduction

The `odes.mac` package by Aleksas Domarkas dates back to 2013. It is very useful and contains many functions for studying or solving differential equations. However, Maxima has evolved considerably since the package was written, and some commands are no longer compatible. In addition, bugs have appeared in several functions. With the help of Claude AI, a new version of this package, called `odes2`, is proposed. It fixes some of these bugs and is compatible with recent versions of Maxima.

This document was generated from the documentation of the original `odes` package and by using `odes2`. The corresponding wxMaxima file can be downloaded from the <https://maxima-french-doc.fr/> website, under the *Examples* section (`manual-odes2.wxmx`).

2 Installation

1. Download the `odes2.mac` file. The file is compressed in ZIP format, so it must be extracted before it can be used.
2. Copy this file into the Maxima folder.
3. Load the package with the command `load(odes2)`.

Alternatively, it can be loaded from the wxMaxima menu: File → Load Package.

3 Commands of the package

<code>odecv</code>	<code>intfactor1</code>
<code>dchange</code>	<code>odeL</code>
<code>odeC</code>	<code>odeL_ic</code>
<code>solv</code>	<code>fs</code>
<code>ode1_ic</code>	<code>partsol</code>
<code>ode2_ic</code>	<code>odeM</code>
<code>P_iter</code>	<code>odeM_ic</code>
<code>odeltaylor</code>	<code>matrix_exp</code>
<code>ode2taylor</code>	<code>odelinsys</code>
<code>odelexact</code>	<code>wronskian</code>

4 Bug Fixes and Improvements in odes2

This revised version of Aleksas Domarkas's `odes.mac` package (original version 2.01, November 2013) preserves the mathematical methods of the original package. The changes mainly concern bug fixes, compatibility with recent versions of Maxima, the reliability of calculations, and improved error messages.

4.1 Argument validation and error messages

Checks have been added to the public functions in order to detect incorrect usage earlier: a missing or malformed equation, an incorrect number of initial conditions, an inappropriate equation order, a non-square matrix, incompatible dimensions, etc.

When a computation cannot be performed, the package now produces an explicit message identifying the function concerned and the nature of the problem. In many cases, this avoids displaying obscure internal Maxima errors.

In particular, the functions `odeL`, `odeL_ic`, `fs`, `icn`, `odeM`, and `odeM_ic` intercept internal errors and replace them with messages that are more useful to the user.

4.2 Linear equations with constant coefficients

The functions `fs`, `odeL`, and `odeL_ic` now check that the supplied equation is linear with constant coefficients. This prevents attempts to solve nonlinear equations or equations with variable coefficients by methods that do not apply to them.

The initial check contained several defects. In particular, `polynomialp` cannot receive derivatives directly as polynomial variables, because expressions such as `'diff(y, x, k)` are not atomic symbols. The derivatives are therefore now replaced, one at a time and in decreasing order, by auxiliary symbols before the linearity test is performed.

Another bug concerned the counting of the roots of the characteristic polynomial in `fs`. The program compared the order of the equation with the number of *distinct* roots, instead of taking their multiplicities into account. The calculation now correctly adds the multiplicities of the roots.

4.3 The `partsol` function

The computation of a particular solution by the method of undetermined coefficients and the Laplace transform has been made more reliable.

In the original version, `partsolL_UC` attempted to extract the particular solution by removing, after expansion, terms assumed to belong to the homogeneous solution. This method relied on a heuristic filter that could fail when `desolve` returned an unexpanded or partially factored expression, especially in the presence of repeated or complex roots.

The revised version instead imposes the zero initial conditions

$$y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0$$

on the auxiliary equation solved by `desolve`. The result is then a well-defined particular solution. It may contain terms that formally belong to the homogeneous solution, but these terms are absorbed into the arbitrary constants of the general solution and therefore do not affect the mathematical validity of the result.

The function `partsol` first tries this method. If it fails, it automatically uses variation of parameters (`partsolL_VP`), without being restricted to a few special forms of the right-hand side.

4.4 Taylor-series functions

The functions `ode1taylor` and `ode2taylor` reused the same local variable to store, successively, the symbolic Taylor series and the output of `solve`. The initially computed series was therefore lost, and the function could return a list of equations instead of the expected series.

This defect has been fixed by separating the two objects: the Taylor series is stored in a dedicated variable, while the solution of the system that determines the coefficients is stored in another variable.

4.5 Exact equations: `ode1exact`

The function `ode1exact` has been corrected for first-order exact differential equations

$$P(x, y) dx + Q(x, y) dy = 0, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The original version delegated an intermediate step to the general-purpose solver `ode2`. After integrating P with respect to x , it was necessary to compute a function $g(y)$ such that

$$g'(y) = Q - \frac{\partial F}{\partial y}.$$

This equation only requires direct integration with respect to y , but `ode2` could return an incorrect coefficient when the unknown function was passed in an auxiliary functional form.

The revised version no longer uses `ode2` in this case. It computes directly

$$F(x, y) = \int P(x, y) dx, \quad g(y) = \int \left(Q(x, y) - \frac{\partial F}{\partial y} \right) dy,$$

and then returns the first integral

$$F(x, y) + g(y) = C.$$

For example, for

$$2xy \, dx + (x^2 + 3y^2) \, dy = 0,$$

the corrected version correctly returns

$$x^2y + y^3 = C.$$

4.6 Management of Maxima's global state

Some functions in the original package temporarily used `depends` or `assume`, but did not always remove the created assumptions or dependencies. If an error occurred during the calculation, they remained active in the Maxima session and could affect subsequent computations.

For example, a residual dependency of the form `depends(y, x)` changes the result of `diff(y, x)` and can subsequently cause functions such as `odelexact` to fail. Similarly, an assumption such as `u>0` or `t>t0` must not persist after a function call.

The functions `odecv`, `odeC`, and `odelinsys` now use `unwind_protect`. Consequently, the cleanup instructions `remove(..., dependency)` and `forget(...)` are executed whether the calculation completes normally or an error occurs.

4.7 The `solv` function

The function `solv`, which solves an algebraic equation and attempts to provide real roots in trigonometric form when necessary, contained an error in its handling of the *casus irreducibilis*.

For an irreducible cubic polynomial with three real roots, `solve` may produce Cardano expressions containing intermediate complex quantities. The former program searched for the occurrence of `sin` in the complete list of roots before carrying out the trigonometric conversion. This test could not correctly detect the case to be treated, because trigonometric functions appear only after that conversion.

The corrected version processes each root separately. It examines the raw root returned by `solve` and then applies, when necessary, the transformations `polarform`, `rectform`, `trigsimp`, and `trigreduce`. This correction makes it possible, in particular, to handle equations such as

$$k^3 - 3k^2 + 1 = 0,$$

whose three roots are real.

4.8 Determination of integration constants

The function `icn`, used in particular by `odeL_ic`, determines the integration constants from the initial conditions. It now checks that the resulting system has exactly one solution.

If the system has no solution, or if it has more than one solution, the function displays an explicit message instead of performing an incorrect or incomplete substitution.

4.9 Regression test suite

A new function, `odes_selftest()`, has been added. It runs a suite of tests covering, in particular:

- simple and repeated real roots;
- pairs of complex conjugate roots;
- repeated complex roots;
- a resonance case for a nonhomogeneous equation;
- a fourth-order initial-value problem;
- Taylor series computed by `odeltaylor`;

- exact equations handled by `odeexact`;
- the absence of leaked dependencies created by `odecv`;
- the *casus irreducibilis* in `solve`.

This test suite makes it easier to verify the package after modifying the code or updating Maxima.

4.10 Known limitations

Some limitations originate in Maxima's native solver and are not fixed in `odes2`.

For some linear equations with constant coefficients whose characteristic polynomial has complicated algebraic roots, `odeL_ic` may still trigger an internal error of type `PQUOTIENT`. This difficulty is related to Maxima's polynomial arithmetic in complicated algebraic extensions.

In addition, the reduction of a second-order autonomous equation to a first-order equation in `ode2_ic` may encounter a known defect in the native `ode2` solver. It is therefore recommended that results from `ode2_ic` and `ode_ic` be independently checked for autonomous equations.

Documentation of odes2

```
(% i1)  /* load the package */
        load(odes2)$
```

5 odecv

Function: `odecv(tr,eq,y,x)` Makes the change of independent variable in ODE.

5.1 $x^3 y''' + x y' - y = x$

```
(% i2)  eq:x^3*'diff(y,x,3)+x*'diff(y,x)-y=x;
```

$$x^3 \left(\frac{d^3}{dx^3} y \right) + x \left(\frac{d}{dx} y \right) - y = x \quad (\text{eq})$$

```
(% i3)  odecv(x=exp(t),eq,y,x);
```

$$\frac{d^3}{dt^3} y - 3 \left(\frac{d^2}{dt^2} y \right) + 3 \left(\frac{d}{dt} y \right) - y = e^t \quad (\% \text{ o3})$$

```
(% i4)  odeL(% ,y,t);
```

$$y = \frac{e^t t^3}{6} + e^t C3 t^2 + e^t C2 t + e^t C1 \quad (\% \text{ o4})$$

```
(% i5)  sol:subst(t=log(x),%);
```

$$y = \frac{x \log(x)^3}{6} + C3 x \log(x)^2 + C2 x \log(x) + C1 x \quad (\text{sol})$$

5.2 $(1 + x^2)y'' + xy' + y = 0$

(% i6) eq:(1+x^2)*diff(y,x,2)+x*diff(y,x)+y=0;

$$(x^2 + 1) \left(\frac{d^2}{dx^2} y \right) + x \left(\frac{d}{dx} y \right) + y = 0 \quad (\text{eq})$$

(% i7) trans(eq):=block(
coeff(lhs(eq),'diff(y,x))/coeff(lhs(eq),'diff(y,x,2)),
t=radcan(integrate(exp(-integrate(%%,x)),x)))\$

(% i9) itr:trans(eq); tr:solve(itr,x)[1];

$$t = \operatorname{asinh}(x) \quad (\text{itr})$$

$$x = \sinh(t) \quad (\text{tr})$$

(% i10) odecv(tr,eq,y,x);

$$\frac{d^2}{dt^2} y + y = 0 \quad (\% \text{ o10})$$

(% i11) ode2(%y,t);

$$y = \%k1 \sin(t) + \%k2 \cos(t) \quad (\% \text{ o11})$$

(% i12) sol:subst(itr,%);

$$y = \%k1 \sin(\operatorname{asinh}(x)) + \%k2 \cos(\operatorname{asinh}(x)) \quad (\text{sol})$$

5.3 $y'' - y'/x + 4x^2y = 0$

(% i13) eq:'diff(y,x,2)-1/x*diff(y,x)+4*x^2*y=0;

$$\frac{d^2}{dx^2} y - \frac{\frac{d}{dx} y}{x} + 4x^2 y = 0 \quad (\text{eq})$$

(% i15) itr:t=x^2; tr:x=sqrt(t);

$$t = x^2 \quad (\text{itr})$$

$$x = \sqrt{t} \quad (\text{tr})$$

(% i16) odecv(t=x^2,eq,y,x);

$$4t \left(\frac{d^2}{dt^2} y \right) + 4ty = 0 \quad (\% \text{ o16})$$

(% i17) ode2(%y,t);

$$y = \%k1 \sin(t) + \%k2 \cos(t) \quad (\% \text{ o17})$$

(% i18) sol:subst(itr,%);

$$y = \%k1 \sin(x^2) + \%k2 \cos(x^2) \quad (\text{sol})$$

5.4 kamke 3.74. $4x^4 y''' - 4x^3 y'' + 4x^2 y' = 1$

(% i19) eq:4*x^4*'diff(y,x,3)-4*x^3*'diff(y,x,2)+4*x^2*'diff(y,x)=1;

$$4x^4 \left(\frac{d^3}{dx^3} y \right) - 4x^3 \left(\frac{d^2}{dx^2} y \right) + 4x^2 \left(\frac{d}{dx} y \right) = 1 \quad (\text{eq})$$

(% i20) odecv(x=exp(t),eq,y,x);

$$4e^t \left(\frac{d^3}{dt^3} y \right) - 16e^t \left(\frac{d^2}{dt^2} y \right) + 16e^t \left(\frac{d}{dt} y \right) = 1 \quad (\% \text{ o20})$$

(% i21) eq1:%/4/exp(t),expand;

$$\frac{d^3}{dt^3} y - 4 \left(\frac{d^2}{dt^2} y \right) + 4 \left(\frac{d}{dt} y \right) = \frac{e^{-t}}{4} \quad (\text{eq1})$$

(% i22) odeL(eq1,y,t);

$$y = \frac{e^{-t} (6e^{3t}t - 5e^{3t} + 9e^t - 4)}{144} + e^{2t}C3t + e^{2t}C2 + C1 \quad (\% \text{ o22})$$

(% i23) subst(t=log(x),%),expand;

$$y = C3x^2 \log(x) + \frac{x^2 \log(x)}{24} + C2x^2 - \frac{5x^2}{144} - \frac{1}{36x} + C1 + \frac{1}{16} \quad (\% \text{ o23})$$

(% i24) y=collectterms(rhs(%),log(x),x);

$$y = \left(C3 + \frac{1}{24} \right) x^2 \log(x) + \left(C2 - \frac{5}{144} \right) x^2 - \frac{1}{36x} + C1 + \frac{1}{16} \quad (\% \text{ o24})$$

(% i25) sol:subst([C1=%k1,C2=%k2,C3=%k3-1/24],%);

$$y = k3x^2 \log(x) + \left(k2 - \frac{5}{144} \right) x^2 - \frac{1}{36x} + k1 + \frac{1}{16} \quad (\text{sol})$$

A comparison with the result given in the original documentation shows that integration constants have been added, but the results are equivalent and correct.

6 dchange

Function: dchange(tr,eq,y,x,new_func,new_var)Makes the change tr:x=f(new_var) of independent variable x.

(% i26) /* these examples need the package contrib_odes\$*/
load(contrib_ode)\$

6.1 $(1 - x^2) y'' - x y' + n^2 y = 0$

(% i27) eq:(1-x^2)*diff(y(x),x,2)-x*diff(y(x),x)+n^2*y(x)=0;

$$(1 - x^2) \left(\frac{d^2}{dx^2} y(x) \right) - x \left(\frac{d}{dx} y(x) \right) + n^2 y(x) = 0 \quad (\text{eq})$$

(% i28) assume(n>0)\$

(% i30) tr:x=cos(t); itr:t=acos(x);

$$x = \cos(t) \quad (\text{tr})$$

$$t = \arccos(x) \quad (\text{itr})$$

(% i32) dchange(tr,eq,y(x),x,y(t),t)\$ eq1:trigsimp(%);

$$\frac{d^2}{dt^2} y(t) + n^2 y(t) = 0 \quad (\text{eq1})$$

(% i33) ode2(%y(t),t);

$$y(t) = \%k1 \sin(nt) + \%k2 \cos(nt) \quad (\% \text{ o33})$$

(% i34) sol:dchange(itr,%,y(t),t,y(x),x);

$$y(x) = \%k1 \sin(n \arccos(x)) + \%k2 \cos(n \arccos(x)) \quad (\text{sol})$$

6.2 $xy'' + y'/2 - y = 0$

(% i35) eq:x*diff(y(x),x,2)+diff(y(x),x)/2-y(x)=0;

$$x \left(\frac{d^2}{dx^2} y(x) \right) + \frac{d}{dx} y(x) - y(x) = 0 \quad (\text{eq})$$

(% i37) tr:x=t^2/4; itr:solve(%,t)[2];

$$x = \frac{t^2}{4} \quad (\text{tr})$$

$$t = 2\sqrt{x} \quad (\text{itr})$$

(% i38) eq1:dchange(tr,eq,y(x),x,y(t),t),ratsimp;

$$\frac{d^2}{dt^2} y(t) - y(t) = 0 \quad (\text{eq1})$$

(% i39) ode2(%y(t),t);

$$y(t) = \%e^{-t}\%k2 + \%e^t \%k1 \quad (\% \text{ o39})$$

(% i40) sol:dchange(itr,%,y(t),t,y(x),x);

$$y(x) = \%e^{-(2\sqrt{x})}\%k2 + \%e^{2\sqrt{x}} \%k1 \quad (\text{sol})$$

6.3 $x^2 y'' - 2xy' + 2y = x^5 \log(x)$

(% i41) eq:x^2*diff(y(x),x,2)-2*x*diff(y(x),x)+2*y(x)=x^5*log(x);

$$x^2 \left(\frac{d^2}{dx^2} y(x) \right) - 2x \left(\frac{d}{dx} y(x) \right) + 2y(x) = x^5 \log(x) \quad (\text{eq})$$

(% i43) tr:x=exp(t); itr:solve(%,t)[1];

$$x = \%e^t \quad (\text{tr})$$

$$t = \log(x) \quad (\text{itr})$$

(% i44) eq1:dchange(tr,eq,y(x),x,y(t),t);

$$\frac{d^2}{dt^2} y(t) - 3 \left(\frac{d}{dt} y(t) \right) + 2 y(t) = e^{5t} t \quad (\text{eq1})$$

(% i45) ode2(%y(t),t);

$$y(t) = \frac{12e^{5t}t - 7e^{5t}}{144} + e^t k_2 + e^{2t} k_1 \quad (\% \text{ o45})$$

(% i46) sol:dchange(itr,%y(t),t,y(x),x);

$$y(x) = \frac{12x^5 \log(x) - 7x^5}{144} + k_1 x^2 + k_2 x \quad (\text{sol})$$

6.4 kamke 2.284

(% i47) eq:(2*x+1)^2*diff(y(x),x,2)-2*(2*x+1)*diff(y(x),x)-12*y(x)=3*x+1;

$$(2x+1)^2 \left(\frac{d^2}{dx^2} y(x) \right) - 2(2x+1) \left(\frac{d}{dx} y(x) \right) - 12 y(x) = 3x+1 \quad (\text{eq})$$

(% i49) tr:x=(e^t-1)/2; itr:solve(tr,t)[1];

$$x = \frac{e^t - 1}{2} \quad (\text{tr})$$

$$t = \log(2x+1) \quad (\text{itr})$$

(% i50) eq1:dchange(tr,eq,y(x),x,y(t),t);

$$4 \left(\frac{d^2}{dt^2} y(t) \right) - 8 \left(\frac{d}{dt} y(t) \right) - 12 y(t) = \frac{3(e^t - 1)}{2} + 1 \quad (\text{eq1})$$

(% i51) ode2(eq1,y(t),t);

$$y(t) = e^{-t} k_2 + e^{3t} k_1 - \frac{9e^t - 4}{96} \quad (\% \text{ o51})$$

(% i52) sol:dchange(itr,%y(t),t,y(x),x);

$$y(x) = k_1(2x+1)^3 + \frac{k_2}{2x+1} - \frac{9(2x+1) - 4}{96} \quad (\text{sol})$$

(% i53) y(x)=map(factor,rhs(sol));

$$y(x) = k_1(2x+1)^3 + \frac{k_2}{2x+1} - \frac{18x+5}{96} \quad (\% \text{ o53})$$

6.5 $y'' - y' + e^{4x} y = 0$

(% i54) eq:diff(y(x),x,2)-diff(y(x),x)+exp(4*x)*y(x)=0;

$$\frac{d^2}{dx^2} y(x) - \frac{d}{dx} y(x) + e^{4x} y(x) = 0 \quad (\text{eq})$$

(% i55) tr:x=log(t)/4;

$$x = \frac{\log(t)}{4} \quad (\text{tr})$$

(% i56) itr:solve(tr,t)[1];

$$t = \%e^{4x} \quad (\text{itr})$$

(% i57) eq1:dchange(tr,eq,y(x),x,y(t),t);

$$16t^2 \left(\frac{d^2}{dt^2} y(t) \right) + 12t \left(\frac{d}{dt} y(t) \right) + t y(t) = 0 \quad (\text{eq1})$$

(% i58) eq2:subst(y(t)=y,eq1);

$$16t^2 \left(\frac{d^2}{dt^2} y \right) + 12t \left(\frac{d}{dt} y \right) + t y = 0 \quad (\text{eq2})$$

(% i59) contrib_ode(eq2,y,t);

$$\left[y = \text{bessel_y} \left(\frac{1}{4}, \frac{\sqrt{t}}{2} \right) \%k2 t^{\frac{1}{8}} + \text{bessel_j} \left(\frac{1}{4}, \frac{\sqrt{t}}{2} \right) \%k1 t^{\frac{1}{8}} \right] \quad (\% \text{ o59})$$

(% i60) sol:subst(itr,%[1]);

$$y = \%e^{\frac{x}{2}} \text{bessel_y} \left(\frac{1}{4}, \frac{\%e^{2x}}{2} \right) \%k2 + \%e^{\frac{x}{2}} \text{bessel_j} \left(\frac{1}{4}, \frac{\%e^{2x}}{2} \right) \%k1 \quad (\text{sol})$$

7 odeC

Function: odeC(eq,r,x)Solves ODE in respect to expression r.

7.1 Bernoulli differential equation

(% i61) eq:'diff(y,x)+2*y/(x+1)=2*sqrt(y)/(x+1);

$$\frac{d}{dx} y + \frac{2y}{x+1} = \frac{2\sqrt{y}}{x+1} \quad (\text{eq})$$

(% i62) odeC(eq,sqrt(y),x);

$$\sqrt{y} = \frac{x}{x+1} + \frac{\%c}{x+1} \quad (\% \text{ o62})$$

(% i63) ode2(eq,y,x);

$$-\log(\sqrt{y} - 1) = \log(x+1) + \%c \quad (\% \text{ o63})$$

7.2 boj 360.

(% i64) eq:x*'diff(y,x,3)-'diff(y,x,2)-x*'diff(y,x)+y=-2*x^3;

$$x \left(\frac{d^3}{dx^3} y \right) - \frac{d^2}{dx^2} y - x \left(\frac{d}{dx} y \right) + y = -(2x^3) \quad (\text{eq})$$

(% i65) odeC(eq,'diff(y,x,2)+y,x);

$$\frac{d^2}{dx^2} y + y = 2y - x^3 + \%cx \quad (\% \text{ o65})$$

(% i66) ode2(%y,x);

$$y = x^3 + (6 - \%c)x + \%e^{-x} \%k2 + \%e^x \%k1 \quad (\% \text{ o66})$$

(% i67) sol:subst(%c=6-%k3,%);

$$y = x^3 + \%k3x + \%e^{-x} \%k2 + \%e^x \%k1 \quad (\text{sol})$$

7.3 sam 4.35.

```
(% i70) eq1:'diff(x,t)=y+z$
      eq2:'diff(y,t)=x+z$
      eq3:'diff(z,t)=x+y$
(% i72) odeC(eq1+eq2+eq3,x+y+z,t)$
      s1:subst(%c=3*C1,%);
```

$$z + y + x = 3e^{2t}C1 \quad (\text{s1})$$

```
(% i74) odeC(eq1-eq2,y-x,t)$
      s2:subst(%c=3*C2,%);
```

$$y - x = 3e^{-t}C2 \quad (\text{s2})$$

```
(% i76) odeC(eq2-eq3,z-y,t)$
      s3:subst(%c=3*C3,%);
```

$$z - y = 3e^{-t}C3 \quad (\text{s3})$$

```
(% i77) sol:solve([s1,s2,s3],[x,y,z])[1],expand;
```

$$\left[x = - (e^{-t}C3) - 2e^{-t}C2 + e^{2t}C1, y = - (e^{-t}C3) + e^{-t}C2 + e^{2t}C1, z = 2e^{-t}C3 + e^{-t}C2 + e^{2t}C1 \right] \quad (\text{sol})$$

Test :

```
(% i80) subst(sol,[eq1,eq2,eq3])$
      ev(% , nouns)$
      makelist(rhs(%[k])-lhs(%[k]),k,1,3);
```

$$[0, 0, 0] \quad (\% \text{ o80})$$

7.4 filipov 65.

```
(% i81) eq:'diff(y,x)=sqrt(4*x+2*y-1);
```

$$\frac{d}{dx}y = \sqrt{2y + 4x - 1} \quad (\text{eq})$$

```
(% i82) ode2(eq,y,x);
```

(% o82)

false

```
(% i84) /*answer of maxima question is positive*/
      assume(p>0);
      contrib_ode(eq,y,x);
```

$$[p > 0] \quad (\% \text{ o83})$$

$$\left[- \left(\frac{(8 \log(\sqrt{2y + 4x - 1} + 2)) + 4\sqrt{2y + 4x - 1} - 4x + 1}{4} \right) = \%c \right] \quad (\% \text{ o84})$$

```
(% i85) odeC(eq,2*y+4*x-1,x);
```

$$- \left(2 \log(\sqrt{2y + 4x - 1} + 2) \right) + \sqrt{2y + 4x - 1} + 2 = x + \%c \quad (\% \text{ o85})$$

8 solvet

Function: `solvet(eq,x)` Returns rectform solution of polynomial equation. In "casus irreducibilis" give real solutions expressed in trigonometric functions. One version of "solvet" is:

```
(% i86) solvet(eq,x):=block([polf,spr,k],
  spr:solve(eq,x),
  polf(x):=block([rx],
    rx:rectform(x),
    if freeof(%i,x) or atom(x) or
    freeof(sin,rx) then return(rx) else
    map(polarform,x),
    rectform(%%),
    trigsimp(%%),
    trigreduce(%%)),
  makelist(x=polf(rhs(spr[k])),k,1,length(spr)),
  sort(%%)
)$
```

Examples

```
(% i87) solvet(x^3-3*x^2+1,x);
```

$$\left[x = 2 \cos\left(\frac{\pi}{9}\right) + 1, x = 2 \cos\left(\frac{5\pi}{9}\right) + 1, x = 2 \cos\left(\frac{7\pi}{9}\right) + 1 \right] \quad (\% \text{ o87})$$

```
(% i88) solvet(x^6-3*x^5-3*x^4+12*x^3-3*x^2-6*x+2=0,x);
```

$$\left[x = 1, x = 1 - \sqrt{3}, x = \sqrt{3} + 1, x = 2 \cos\left(\frac{2\pi}{9}\right), x = 2 \cos\left(\frac{4\pi}{9}\right), x = 2 \cos\left(\frac{8\pi}{9}\right) \right] \quad (\% \text{ o88})$$

```
(% i89) solvet(x^3-15*x-5,x);
```

$$\left[x = 2\sqrt{5} \cos\left(\frac{\operatorname{atan}\left(\frac{\sqrt{19}}{3}\right) - \frac{2\pi}{3}}{3}\right), x = 2\sqrt{5} \cos\left(\frac{\operatorname{atan}\left(\frac{\sqrt{19}}{3}\right) + \frac{2\pi}{3}}{3}\right), x = 2\sqrt{5} \cos\left(\frac{\operatorname{atan}\left(\frac{\sqrt{19}}{3}\right)}{3}\right) \right] \quad (\% \text{ o89})$$

```
(% i90) solve(x^3-15*x-5,x);
```

$$\left[x = \left(- \left(\frac{\sqrt{3}\%i}{2} \right) - \frac{1}{2} \right) \left(\frac{5\sqrt{19}\%i}{2} + \frac{5}{2} \right)^{\frac{1}{3}} + \frac{5 \left(\frac{\sqrt{3}\%i}{2} - \frac{1}{2} \right)}{\left(\frac{5\sqrt{19}\%i}{2} + \frac{5}{2} \right)^{\frac{1}{3}}}, x = \left(\frac{\sqrt{3}\%i}{2} - \frac{1}{2} \right) \left(\frac{5\sqrt{19}\%i}{2} + \frac{5}{2} \right)^{\frac{1}{3}} + \frac{5 \left(- \left(\frac{\sqrt{3}\%i}{2} \right) \right)}{\left(\frac{5\sqrt{19}\%i}{2} + \frac{5}{2} \right)^{\frac{1}{3}}} \right] \quad (\% \text{ o90})$$

9 ode1_ic

Function: `ode1_ic(eqn, dvar, ivar, ic)` The function `ode1_ic` solves an ordinary differential equation (ODE) of first order with initial condition $y(x_0) = y_0$. Here `ic` is list $[x_0, y_0]$. Examples :

9.1 $x^2 y' + 3xy = \sin(x)/x, y(\pi) = 0$

```
(% i91) ode1_ic(x^2*'diff(y,x) + 3*y*x = sin(x)/x,y,x,[%pi,0]);
```

$$y = - \left(\frac{\cos(x) + 1}{x^3} \right) \quad (\% \text{ o91})$$

$$9.2 \quad (y^4 * \%e^y + 2 * x) * y' = y, \quad y(0) = 1$$

(% i92) eq:(y^4*exp(y)+2*x)*diff(y,x)=y;

$$(\%e^y y^4 + 2x) \left(\frac{d}{dx} y \right) = y \quad (\text{eq})$$

(% i93) ode1_ic(eq,y,x,[0,1]);

$$\frac{\%e^y y^3 - \%e^y y^2 - x}{y^2} = 0 \quad (\% \text{ o93})$$

(% i94) solve(% ,x);

$$[x = \%e^y y^3 - \%e^y y^2] \quad (\% \text{ o94})$$

$$9.3 \quad x * y' + y = 2 * y^2 * \log(x), \quad y(1) = 1/2$$

(% i95) eq:x*diff(y,x)+y=2*y^2*log(x);

$$x \left(\frac{d}{dx} y \right) + y = 2 \log(x) y^2 \quad (\text{eq})$$

(% i96) ode1_ic(eq,y,x,[1,1/2]);

$$y = \frac{1}{2 \log(x) + 2} \quad (\% \text{ o96})$$

$$9.4 \quad (x^2 - 1) * y' + 2 * x * y^2 = 0, \quad y(0) = 1$$

(% i97) eq:(x^2-1)*diff(y,x)+2*x*y^2=0\$

(% i98) ode1_ic(eq,y,x,[0,1]);

$$y = \frac{1}{\log(1 - x^2) + 1} \quad (\% \text{ o98})$$

10 ode2_ic

Function: ode2_ic(eqn, dvar, ivar, ic) The function ode2_ic solve an ordinary differential equation(ODE) of second order with initial conditions $y(x_0) = y_0$, $y'(x_0) = y_1$. Here ic is list $[x_0, y_0, y_1]$. Examples :

$$10.1 \quad y'' + y * y'^3 = 0, \quad y(0) = 0, \quad y'(0) = 2$$

(% i99) eq:'diff(y,x,2) + y*'diff(y,x)^3 = 0;

$$\frac{d^2}{dx^2} y + y \left(\frac{d}{dx} y \right)^3 = 0 \quad (\text{eq})$$

(% i100) sol:ode2_ic(eq,y,x,[0,0,2]);

$$y = \left(\sqrt{9x^2 + 1} + 3x \right)^{\frac{1}{3}} - \frac{1}{\left(\sqrt{9x^2 + 1} + 3x \right)^{\frac{1}{3}}} \quad (\text{sol})$$

Test

```
(% i101) ev(rhs(sol),x=0);
```

0

(% o101)

```
(% i103) diff(rhs(sol),x)$ ev(%,x=0);
```

2

(% o103)

```
(% i106) subst(sol,eq)$
ev(%, nouns)$
radcan(%);
```

0 = 0

(% o106)

10.2 $y'' = 128y^3$, $y(0) = 1$, $y'(0) = 8$

```
(% i107) eq:'diff(y,x,2)=128*y^3;
```

$$\frac{d^2}{dx^2}y = 128y^3$$

(eq)

-> /* Due to a bug in odes, this example leads to an error message
the bug is due to a bug in Maxima itself, so it is impossible to correct
ode2_ic(eq,y,x,[0,1,8]);
the command is put in comment*/;

10.3 $y'' + y = 1/\cos(x)$, $y(0)=1$, $y'(0)=0$.

```
(% i108) eq:'diff(y,x,2)+y = 1/cos(x);
```

$$\frac{d^2}{dx^2}y + y = \frac{1}{\cos(x)}$$

(eq)

```
(% i109) sol:ode2_ic(eq,y,x,[0,1,0]);
```

$$y = \cos(x) \log(\cos(x)) + x \sin(x) + \cos(x)$$

(sol)

11 P_iter

Function: P_iter(eq, x, y, x0, y0, n).Solves first order differential equation using Picard iterative process.(https://www.ams.stonybrook.edu/~jiao/teaching/ams501_fall11/notes/picard.pdf)Examples :

11.1 $y' = x^2 + y^2$, $y(0) = 0$

```
(% i110) eq:'diff(y,x)=x^2+y^2;
```

$$\frac{d}{dx}y = y^2 + x^2$$

(eq)

```
(% i112) x0:0$ y0:0$
```

```
(% i113) for k:0 thru 3 do
print(P_iter(eq,x,y,x0,y0,k))$
```

$$0 \frac{x^3}{3} \frac{x^7}{63} + \frac{x^3}{3} \frac{x^{15}}{59535} + \frac{2x^{11}}{2079} + \frac{x^7}{63} + \frac{x^3}{3}$$

11.2 $y' = 2x(1 + y), y(0) = 0.$

(% i114) eq:'diff(y,x)=2*x*(1+y);

$$\frac{d}{dx}y = 2x (y + 1) \quad (\text{eq})$$

(% i116) x0:0\$ y0:0\$

(% i117) for k:0 thru 5 do

print(P_iter(eq,x,y,x0,y0,k))\$

$$0 \quad x^2 \frac{x^4}{2} + x^2 \frac{x^6}{6} + \frac{x^4}{2} + x^2 \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2 \frac{x^{10}}{120} + \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2$$

12 ode1taylor

Function: ode1taylor(eq, x0, y0, n).Solves first order differential equation using Taylor-series expansion.ode1taylor was buggy in the original odes package. The bugs are solved in this version of odes2

12.1 $y' = x + y^2, y(0) = 1$

(% i118) eq:diff(y(x),x)=x+y(x)^2;

$$\frac{d}{dx}y(x) = y(x)^2 + x \quad (\text{eq})$$

(% i119) ode1taylor(eq,0,1,6);

$$1 + x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{17x^4}{12} + \frac{31x^5}{20} + \frac{149x^6}{90} + \dots \quad (\% \text{ o119/T/})$$

12.2 $y' = x^2 + y^2, y(0) = 0$

(% i120) eq:diff(y(x),x)=x^2+y(x)^2;

$$\frac{d}{dx}y(x) = y(x)^2 + x^2 \quad (\text{eq})$$

(% i121) ode1taylor(eq,0,0,15);

$$\frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{13x^{15}}{218295} + \dots \quad (\% \text{ o121/T/})$$

12.3 $y' = x - y^2, y(1) = -1$

(% i122) eq:diff(y(x),x)=x-y(x)^2;

$$\frac{d}{dx}y(x) = x - y(x)^2 \quad (\text{eq})$$

(% i123) ode1taylor(eq,1,-1,5);

$$-1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{6} + \frac{(x-1)^5}{60} + \dots \quad (\% \text{ o123/T/})$$

13 ode2taylor (bug corrected from original odes)

Function: ode2taylor(eq, x0, y0, y1, n).Solves second order differential equation using Taylor-series expansion.ode2taylor was buggy in the original odes package. The bugs are solved in this version of odes2Examples

13.1 Airy's Equation $y''-xy=0$, $y(0)=1$, $y'(0)=0$.

(% i124) eq:'diff(y(x),x,2)-x*y(x)=0;

$$\frac{d^2}{dx^2} y(x) - x y(x) = 0 \quad (\text{eq})$$

(% i125) ode2taylor(eq,0,1,0,15);

$$1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \frac{x^{12}}{1710720} + \frac{x^{15}}{359251200} + \dots \quad (\% \text{ o125/T/})$$

13.2 $y'' = (y')^2 + x*y$, $y(1) = 1$, $y'(0) = 0$

(% i126) eq:'diff(y(x),x,2)='diff(y(x),x,1)^2+x*y(x);

$$\frac{d^2}{dx^2} y(x) = \left(\frac{d}{dx} y(x) \right)^2 + x y(x) \quad (\text{eq})$$

(% i127) ode2taylor(eq,1,1,0,5);

$$1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{8} + \frac{(x-1)^5}{12} + \dots \quad (\% \text{ o127/T/})$$

13.3 $y''+xy+y=0$

(% i128) eq:'diff(y(x),x,2)+x*'diff(y(x),x)+y(x)=0;

$$\frac{d^2}{dx^2} y(x) + x \left(\frac{d}{dx} y(x) \right) + y(x) = 0 \quad (\text{eq})$$

(% i129) ode2taylor(eq,0,0,1,15);

$$x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - \frac{x^{11}}{10395} + \frac{x^{13}}{135135} - \frac{x^{15}}{2027025} + \dots \quad (\% \text{ o129/T/})$$

(% i130) sum((-1)^n*2^n*n!*x^(2*n+1)/(2*n+1)!,n,0,7);

$$- \left(\frac{x^{15}}{2027025} \right) + \frac{x^{13}}{135135} - \frac{x^{11}}{10395} + \frac{x^9}{945} - \frac{x^7}{105} + \frac{x^5}{15} - \frac{x^3}{3} + x \quad (\% \text{ o130})$$

14 ode1exact

Function: ode1exact(eq).Solves first order exact equation. <https://tutorial.math.lamar.edu/classes/de/exact.aspx>Examples :

(% i131) kill(all);

done (% o0)

(% i1) load(odes2)\$

The solutions of the following equations are false with the original odes.mac. It is due to a bug in ode2. The new version odes2 correct this problem (solving directly without ode2) and so the solutions are now correct.

14.1 $2xydx + (x^2 + 3y^2)dy = 0$

Bug in ode2 so that the solution is false. The new version of odes correct this problem (solving directly without ode2)

(% i2) `eq:2*x*y*dx+(x^2+3*y^2)*dy=0;`

$$dy (3y^2 + x^2) + 2dxx y = 0 \quad (\text{eq})$$

(% i3) `ode1exact(eq);`

$$y^3 + x^2 y = C \quad (\% \text{ o3})$$

14.2 $(6x^2 - y + 3)dx + (3y^2 - x - 3)dy = 0$

(% i4) `eq:(6*x^2-y+3)*dx+(3*y^2-x-3)*dy=0;`

$$dy (3y^2 - x - 3) + dx (-y + 6x^2 + 3) = 0 \quad (\text{eq})$$

(% i5) `ode1exact(eq);`

$$y^3 - xy - 3y + 2x^3 + 3x = C \quad (\% \text{ o5})$$

14.3 $\exp(y)dx + (2y + x\exp(y))dy = 0$

(% i6) `eq:exp(y)*dx+(2*y+x*exp(y))*dy=0;`

$$dy (2y + e^y x) + e^y dx = 0 \quad (\text{eq})$$

(% i7) `ode1exact(eq);`

$$y^2 + e^y x = C \quad (\% \text{ o7})$$

14.4 $(x dx + y dy) / \sqrt{x^2 + y^2} + (x dy - y dx) / x^2 = 0$

(% i8) `eq:(x*dx+y*dy)/sqrt(x^2+y^2)+(x*dy-y*dx)/x^2=0;`

$$\frac{dy y + dx x}{\sqrt{y^2 + x^2}} + \frac{dy x - dx y}{x^2} = 0 \quad (\text{eq})$$

(% i9) `ode1exact(eq);`

$$\sqrt{y^2 + x^2} + \frac{y}{x} = C \quad (\% \text{ o9})$$

15 `intfactor1`

Function: `intfactor(eq, omega)`. Find `intfactor mu = mu(omega)` of the first order differential equation. [https://math.libretexts.org/Bookshelves/Differential_Equations/Differential_Equations_for_Engineers_\(Lebl\)/1:_First_order_ODEs/1.8:_Exact_Equations](https://math.libretexts.org/Bookshelves/Differential_Equations/Differential_Equations_for_Engineers_(Lebl)/1:_First_order_ODEs/1.8:_Exact_Equations)

15.1 (1 + y^2)*dx + x*y*dy = 0

(% i10) eq:(1+y^2)*dx+x*y*dy=0;

$$dx (y^2 + 1) + dy xy = 0 \quad (\text{eq})$$

(% i11) intfactor1(eq,x);

$$x \quad (\% \text{ o11})$$

(% i12) odelexact(eq*%);

$$\frac{x^2 y^2}{2} + \frac{x^2}{2} = C \quad (\% \text{ o12})$$

15.2 (x*y^2 - 2*y^3)*dx + (3 - 2*x*y^2)*dy = 0

(% i13) eq:(x*y^2-2*y^3)*dx+(3-2*x*y^2)*dy=0;

$$dx (xy^2 - 2y^3) + dy (3 - 2xy^2) = 0 \quad (\text{eq})$$

(% i14) intfactor1(eq,y);

$$\frac{1}{y^2} \quad (\% \text{ o14})$$

(% i15) odelexact(eq*%);

$$-(2xy) - \frac{3}{y} + \frac{x^2}{2} = C \quad (\% \text{ o15})$$

15.3 y*dx + (x^2 + y^2 - x)*dy = 0

(% i16) eq:y*dx+(x^2+y^2-x)*dy=0;

$$dy (y^2 + x^2 - x) + dx y = 0 \quad (\text{eq})$$

(% i17) odelexact(eq);

(% o17)

false

(% i18) intfactor1(eq,x^2+y^2);

$$\frac{1}{y^2 + x^2} \quad (\% \text{ o18})$$

(% i19) odelexact(eq*%);

$$y + \operatorname{atan}\left(\frac{x}{y}\right) = C \quad (\% \text{ o19})$$

$$15.4 \quad x*y*dx + (2*x^2 + 3*y^2 - 20)*dy = 0$$

(% i20) eq:x*y*dx+(2*x^2+3*y^2-20)*dy=0;

$$dy (3y^2 + 2x^2 - 20) + dxxy = 0 \quad (\text{eq})$$

(% i21) intfactor1(eq,y);

$$y^3 \quad (\% \text{ o21})$$

(% i22) ode1exact(eq*%);

$$\frac{y^6}{2} + \frac{x^2y^4}{2} - 5y^4 = C \quad (\% \text{ o22})$$

$$15.5 \quad (x^2*y^3 + 6*y^5)*dx + (2*x^3*y^2 + 12*x^4)*dy = 0$$

(% i23) eq:(x^2*y^3+6*y^5)*dx+(2*x^3*y^2+12*x^4)*dy=0;

$$dx (6y^5 + x^2y^3) + dy (2x^3y^2 + 12x^4) = 0 \quad (\text{eq})$$

(% i24) intfactor1(eq/y,x*y);

$$\frac{1}{x^4y^4} \quad (\% \text{ o24})$$

(% i25) ode1exact(-eq/y*%);

$$\frac{1}{xy^2} + \frac{3}{y^4} + \frac{2}{x^3} = C \quad (\% \text{ o25})$$

Other method:

(% i26) mu:x^a*y^b;

$$x^a y^b \quad (\text{mu})$$

(% i27) diff(mu*(x^2*y^3+6*y^5),y)=diff(mu*(2*x^3*y^2+12*x^4),x)\$

(% i28) factor(lhs(%)-rhs(%));

$$x^a y^b (6by^4 + 30y^4 + bx^2y^2 - 2ax^2y^2 - 3x^2y^2 - 12ax^3 - 48x^3) \quad (\% \text{ o28})$$

(% i29) collectterms(%/mu,x,y);

$$(6b + 30)y^4 + (b - 2a - 3)x^2y^2 + (-12a - 48)x^3 \quad (\% \text{ o29})$$

(% i30) solve([coeff(%,y^4),coeff(%,x^2*y^2),coeff(%,x^3)],[a,b]);

solve: dependent equations eliminated: (2)

$$[[a = -4, b = -5]] \quad (\% \text{ o30})$$

(% i31) 'mu=subst(%[1],mu);

$$\mu = \frac{1}{x^4y^5} \quad (\% \text{ o31})$$

16 odeL

Function: odeL(eqn, dvar, ivar) The function odeL solves an linear ODEs with constant coefficients. Examples :

```
(% i32) load(contrib_ode)$
```

16.1 $y''' - 2y'' + y' = 0$.

```
(% i33) eq:'diff(y,x,3)-2*'diff(y,x,2)+'diff(y,x) = 0;
```

$$\frac{d^3}{dx^3}y - 2 \left(\frac{d^2}{dx^2}y \right) + \frac{d}{dx}y = 0 \quad (\text{eq})$$

```
(% i34) odeL(eq,y,x);
```

$$y = \%e^x C3x + \%e^x C2 + C1 \quad (\% \text{ o34})$$

16.2 $y'''' + 8y'' + 16y = x \exp(3x) + \sin(x)^2 + 1$

```
(% i35) eq:'diff(y,x,4)+8*'diff(y,x,2)+16*y=x*exp(3*x)+sin(x)^2+1;
```

$$\frac{d^4}{dx^4}y + 8 \left(\frac{d^2}{dx^2}y \right) + 16y = \sin(x)^2 + \%e^{3x}x + 1 \quad (\text{eq})$$

```
(% i36) sol:odeL(eq,y,x);
```

$$y = - \left(\frac{(26065x - 1992) \sin(2x) + (- (4394x^2) + 1040x + 24828) \cos(2x) - 1664\%e^{3x}x + 1536\%e^{3x} - 26364}{281216} \right) + C4 \quad (\text{sol})$$

```
(% i37) ode_check(eq,sol);
```

$$0 \quad (\% \text{ o37})$$

16.3 $y''' - 3y'' + y = \sin(x)^3$

```
(% i38) eq:'diff(y,x,3)-3*'diff(y,x,2)+y=sin(x)^3;
```

$$\frac{d^3}{dx^3}y - 3 \left(\frac{d^2}{dx^2}y \right) + y = \sin(x)^3 \quad (\text{eq})$$

```
(% i39) /* solvet was buggy, corrected in odes2)*/
solvet(k^3-3*k^2+1=0,k);
```

$$\left[k = 2 \cos\left(\frac{\pi}{9}\right) + 1, k = 2 \cos\left(\frac{5\pi}{9}\right) + 1, k = 2 \cos\left(\frac{7\pi}{9}\right) + 1 \right] \quad (\% \text{ o39})$$

```
-> /* odeL depends of ode2, which uses here Laplace. This bug means that the returned result is incorrect.
sol:odeL(eq,y,x); */;
```

17 odeL_ic

Function: odeL_ic(eqn, dvar, ivar, ic) The function odeL_ic solves initial value problems for linear ODEs with constant coefficients. Examples :

17.1 $y''' + y'' = x + \exp(-x), y(0) = 1, y'(0) = 0, y''(0) = 1.$

(% i40) eq:'diff(y,x,3)+diff(y,x,2)=x + exp(-x);

$$\frac{d^3}{dx^3}y + \frac{d^2}{dx^2}y = x + \%e^{-x} \quad (\text{eq})$$

(% i41) odeL_ic(eq, y, x, [0, 1, 0, 1]),expand;

$$y = \frac{x^3}{6} - \frac{x^2}{2} + \%e^{-x}x + 3x + 4\%e^{-x} - 3 \quad (\% \text{ o41})$$

17.2 $y'''' - y = 8 * \exp(x), y(0)=0, y'(0)=2, y''(0)=4, y'''(0)=6$

(% i42) eq:'diff(y,x,4)-y=8*exp(x);

$$\frac{d^4}{dx^4}y - y = 8\%e^x \quad (\text{eq})$$

(% i43) sol:odeL_ic(eq, y, x, [0, 0, 2, 4, 6]);

$$y = \%e^{-x} \left(2\%e^x \sin(x) + 2\%e^x \cos(x) + 2\%e^{2x}x - 3\%e^{2x} + 1 \right) - 2 \sin(x) - 2 \cos(x) + 3\%e^x - \%e^{-x} \quad (\text{sol})$$

(% i44) expand(sol);

$$y = 2\%e^x x \quad (\% \text{ o44})$$

17.3 $y'''' - y' = 0, y(0)=0, y'(0)=1, y''(0)=0, y'''(0)=1, y''''(0)=2.$

(% i45) eq:'diff(y,x,5)-diff(y,x)=0;

$$\frac{d^5}{dx^5}y - \frac{d}{dx}y = 0 \quad (\text{eq})$$

(% i46) sol:odeL_ic(eq,y,x,[0,0,1,0,1,2]);

$$y = \cos(x) + \%e^x - 2 \quad (\text{sol})$$

Test

(% i47) ode_check(eq,sol);

$$0 \quad (\% \text{ o47})$$

(% i49) makelist(diff(rhs(sol),x,k),k,0,4)\$
ev(% ,x=0);

$$[0, 1, 0, 1, 2] \quad (\% \text{ o49})$$

18 fs

Function: fs(eq, y, x).Find fundamental system of solutions of the linear n-th order differential equation with constant coefficients.

18.1 Example 1

```
(% i50) kill(all)$
(% i1) load(odes2)$
(% i2) eq:'diff(y,x,3)+3*'diff(y,x,2)-10*'diff(y,x)=x-1;
```

$$\frac{d^3}{dx^3}y + 3 \left(\frac{d^2}{dx^2}y \right) - 10 \left(\frac{d}{dx}y \right) = x - 1 \quad (\text{eq})$$

```
(% i3) fs(eq,y,x);
```

$$\left[1, \%e^{-(5x)}, \%e^{2x} \right] \quad (\% \text{ o3})$$

```
(% i4) sol:odeL(eq,y,x);
```

$$y = - \left(\frac{\%e^{-(5x)} \left(350\%e^{5x}x^2 - 490\%e^{5x}x + 125\%e^{7x} - 77\%e^{5x} - 48 \right)}{7000} \right) + \%e^{2x}C3 + \%e^{-(5x)}C2 + C1 \quad (\text{sol})$$

```
(% i5) expand(sol);
```

$$y = - \left(\frac{x^2}{20} \right) + \frac{7x}{100} + \%e^{2x}C3 + \%e^{-(5x)}C2 + C1 - \frac{\%e^{2x}}{56} + \frac{6\%e^{-(5x)}}{875} + \frac{11}{1000} \quad (\% \text{ o5})$$

18.2 Example 2

```
(% i6) eq:'diff(y,x,4)+8*'diff(y,x,2)+16*y=x^2*exp(x)*sin(x);
```

$$\frac{d^4}{dx^4}y + 8 \left(\frac{d^2}{dx^2}y \right) + 16y = \%e^x x^2 \sin(x) \quad (\text{eq})$$

```
(% i7) fs(eq,y,x);
```

$$[\cos(2x), x \cos(2x), \sin(2x), x \sin(2x)] \quad (\% \text{ o7})$$

```
(% i8) sol:odeL(eq,y,x);
```

$$y = - \left(((560x + 2231) \sin(2x) + (208 - 670x) \cos(2x) + (- (600\%e^x x^2) + 2080\%e^x x - 2144\%e^x) \sin(x) + (800\%e^x \right) \quad (\text{sol})$$

18.3 Example 3

```
(% i9) eq:'diff(y,x,8)+'diff(y,x,2)=x^5;
```

$$\frac{d^8}{dx^8}y + \frac{d^2}{dx^2}y = x^5 \quad (\text{eq})$$

```
(% i10) solvet(k^8+k^2=0,k);
```

$$\left[k = 0, k = - \left(\frac{\%i}{2} \right) - \frac{\sqrt{3}}{2}, k = \frac{\sqrt{3}}{2} - \frac{\%i}{2}, k = \frac{\%i}{2} - \frac{\sqrt{3}}{2}, k = \frac{\%i}{2} + \frac{\sqrt{3}}{2}, k = -\%i, k = \%i \right] \quad (\% \text{ o10})$$

```
(% i11) fs(eq,y,x);
```

$$\left[1, \%e^{-\left(\frac{\sqrt{3}x}{2}\right)} \cos\left(\frac{x}{2}\right), \%e^{\frac{\sqrt{3}x}{2}} \cos\left(\frac{x}{2}\right), \%e^{-\left(\frac{\sqrt{3}x}{2}\right)} \sin\left(\frac{x}{2}\right), \%e^{\frac{\sqrt{3}x}{2}} \sin\left(\frac{x}{2}\right), x, \cos(x), \sin(x) \right] \quad (\% \text{ o11})$$

(% i12) sol:odeL(eq,y,x);

$$y = \frac{1680 \sin(x) + x^7 - 5040x + 42 \operatorname{ilt}\left(\frac{80g^{10089^2} - 40}{g^{10089^4} - g^{10089^2} + 1}, g^{10089}, x\right)}{42} + C8 \sin(x) + C7 \cos(x) + C6x + \%e^{\frac{\sqrt{3}x}{2}} C5 \sin(x) \quad (\text{sol})$$

18.4 Example 4

(% i13) eq:'diff(y,x,6)-3*'diff(y,x,5)-3*'diff(y,x,4)+12*'diff(y,x,3)-3*'diff(y,x,2)-6*'diff(y,x,1)+2*y=2*x^7+sin(x)^3;

$$\frac{d^6}{dx^6}y - 3\left(\frac{d^5}{dx^5}y\right) - 3\left(\frac{d^4}{dx^4}y\right) + 12\left(\frac{d^3}{dx^3}y\right) - 3\left(\frac{d^2}{dx^2}y\right) - 6\left(\frac{d}{dx}y\right) + 2y = \sin(x)^3 + 2x^7 \quad (\text{eq})$$

(% i14) fs(eq,y,x);

$$\left[\%e^x, \%e^{2\cos\left(\frac{2\pi}{9}\right)x}, \%e^{2\cos\left(\frac{4\pi}{9}\right)x}, \%e^{2\cos\left(\frac{8\pi}{9}\right)x}, \%e^{x-\sqrt{3}x}, \%e^{\sqrt{3}x+x}\right] \quad (\% \text{ o14})$$

(% i15) sol:odeL(eq,y,x)\$

(% i16) expand(%);

$$y = -\left(\frac{28535483^{\frac{3}{2}}\%e^x \sinh(\sqrt{3}x)}{2041}\right) + \frac{14827640\%e^x \cosh(\sqrt{3}x)}{2041} + \frac{943 \sin(3x)}{8145160} - \frac{1071 \cos(3x)}{8145160} + \frac{3 \sin(x)}{1768} + \frac{63 \cos(x)}{1768} \quad (\% \text{ o16})$$

19 partsol

Function: partsol(eq, y, x). Find partial solution of the linear n-th order differential equation with constant coefficients.

(% i17) kill(all)\$

(% i1) load(odes2)\$

19.1 Example 1

(% i2) eq:'diff(y,x,3)+3*'diff(y,x,2)-10*'diff(y,x)=x-1;

$$\frac{d^3}{dx^3}y + 3\left(\frac{d^2}{dx^2}y\right) - 10\left(\frac{d}{dx}y\right) = x - 1 \quad (\text{eq})$$

(% i3) partsol(eq,y,x);

$$-\left(\frac{\%e^{-(5x)}\left(350\%e^{5x}x^2 - 490\%e^{5x}x + 125\%e^{7x} - 77\%e^{5x} - 48\right)}{7000}\right) \quad (\% \text{ o3})$$

(% i4) expand(%);

$$-\left(\frac{x^2}{20}\right) + \frac{7x}{100} - \frac{\%e^{2x}}{56} + \frac{6\%e^{-(5x)}}{875} + \frac{11}{1000} \quad (\% \text{ o4})$$

(% i5) odeL(eq,y,x);

$$y = -\left(\frac{\%e^{-(5x)}\left(350\%e^{5x}x^2 - 490\%e^{5x}x + 125\%e^{7x} - 77\%e^{5x} - 48\right)}{7000}\right) + \%e^{2x}C3 + \%e^{-(5x)}C2 + C1 \quad (\% \text{ o5})$$

19.2 Example 2

(% i6) eq:'diff(y,x,4)+'diff(y,x,3)-3*'diff(y,x,2)-5*'diff(y,x)-2*y=exp(2*x)-exp(-x);

$$\frac{d^4}{dx^4}y + \frac{d^3}{dx^3}y - 3\left(\frac{d^2}{dx^2}y\right) - 5\left(\frac{d}{dx}y\right) - 2y = e^{2x} - e^{-x} \quad (\text{eq})$$

(% i7) partsol(eq,y,x);

$$\frac{e^{-x} \left(9x^3 + 18x^2 + \left(6e^{3x} + 18 \right) x - 8e^{3x} + 8 \right)}{162} \quad (\% \text{ o7})$$

(% i8) odeL(eq,y,x);

$$y = \frac{e^{-x} \left(9x^3 + 18x^2 + \left(6e^{3x} + 18 \right) x - 8e^{3x} + 8 \right)}{162} + e^{-x} C4x^2 + e^{-x} C3x + e^{2x} C2 + e^{-x} C1 \quad (\% \text{ o8})$$

(% i9) expand(%)\$

(% i10) y=collectterms(rhs(%),exp(-x),exp(2*x));

$$y = e^{-x} \left(\frac{x^3}{18} + C4x^2 + \frac{x^2}{9} + C3x + \frac{x}{9} + C1 + \frac{4}{81} \right) + e^{2x} \left(\frac{x}{27} + C2 - \frac{4}{81} \right) \quad (\% \text{ o10})$$

19.3 Example 3

(% i11) eq:'diff(y,x,3)+'diff(y,x,1)=1/cos(x);

$$\frac{d^3}{dx^3}y + \frac{d}{dx}y = \frac{1}{\cos(x)} \quad (\text{eq})$$

(% i12) partsol(eq,y,x);

$$\text{ilt} \left(\frac{\text{laplace} \left(\frac{1}{\cos(x)}, x, g11444 \right)}{g11444^3 + g11444}, g11444, x \right) \quad (\% \text{ o12})$$

There is a bug in Maxima Laplace integration ...

19.4 Example 4

(% i13) eq:'diff(y,x,3)+8*'diff(y,x,1)+9*y=cos(x)^3;

$$\frac{d^3}{dx^3}y + 8\left(\frac{d}{dx}y\right) + 9y = \cos^3(x) \quad (\text{eq})$$

(% i14) partsol(eq,y,x);

$$- \left((e^{-x} (1080e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{35}x}{2}\right) + 696\sqrt{35}e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{35}x}{2}\right) + 143\sqrt{35}e^x \sin(3x) - 429\sqrt{35}e^x \cos(3x) - 69 \right) \right) \quad (\% \text{ o14})$$

(% i15) sol:odeL(eq,y,x);

$$y = - \left((e^{-x} (1080e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{35}x}{2}\right) + 696\sqrt{35}e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{35}x}{2}\right) + 143\sqrt{35}e^x \sin(3x) - 429\sqrt{35}e^x \cos(3x) - 69 \right) \right) \quad (\text{sol})$$

Test

(% i16) load(contrib_ode)\$

(% i17) ode_check(eq,sol);

0

(% o17)

20 odeM

Function: odeM(A,F,t) Find solutions of linear system of ODEs with constant coefficients in matrix form: $Y' = AY + F$

20.1 Example 1 : $Y' = AY + F$.

```
(% i18) kill(all)$
(% i1)  load(odes2)$
(% i2)  A:matrix([1,1],[4,1]);
```

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \quad (\text{A})$$

```
(% i3)  F:transpose([t-2,4*t-1]);
```

$$\begin{bmatrix} t-2 \\ 4t-1 \end{bmatrix} \quad (\text{F})$$

```
(% i4)  sol:odeM(A,F,t);
```

$$\begin{bmatrix} -t + \left(\frac{\%e^{3t}}{4} - \frac{\%e^{-t}}{4}\right) C2 + \left(\frac{\%e^{3t}}{2} + \frac{\%e^{-t}}{2}\right) C1 \\ \left(\frac{\%e^{3t}}{2} + \frac{\%e^{-t}}{2}\right) C2 + \left(\%e^{3t} - \%e^{-t}\right) C1 + 1 \end{bmatrix} \quad (\text{sol})$$

Test

```
(% i6)  diff(sol,t)-A.sol-F$
        expand(%);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\% \text{ o6})$$

20.2 Example 2 : $Y' = AY$.

```
(% i7)  A:matrix([2,0,-8,-3],[-18,-1,0,0],[-9,-3,-25,-9],[33,10,90,32]);
```

$$\begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} \quad (\text{A})$$

```
(% i8)  F:transpose([0,0,0,0])$
```

```
(% i9)  charpoly(A, x),factor;
```

$$(x^2 - 4x + 13)^2 \quad (\% \text{ o9})$$

```
(% i10) solve(%);
```

$$[x = 2 - 3\%i, x = 3\%i + 2] \quad (\% \text{ o10})$$

```
(% i11) odeM(A,F,t)$
```

```
(% i12) sol:ratsimp(%);
```

$$\begin{bmatrix} \left(\left(- \left(3\%e^{2t} C4 \right) - 9\%e^{2t} C3 - \%e^{2t} C2 - 3\%e^{2t} C1 \right) t - \%e^{2t} C4 - 3\%e^{2t} C3 \right) \sin(3t) + \left(\left(9\%e^{2t} C4 + 24\%e^{2t} C3 + 3\%e^{2t} C2 + 9\%e^{2t} C1 \right) t + 3\%e^{2t} C4 + 10\%e^{2t} C3 - 3\%e^{2t} C1 \right) \sin(3t) + \left(- \left(9\%e^{2t} C4 \right) - \left(- \left(3\%e^{2t} C4 \right) - 9\%e^{2t} C3 - \%e^{2t} C2 - 3\%e^{2t} C1 \right) \sin(3t) + \%e^{2t} C3 \right) \sin(3t) + \left(\%e^{2t} C3 t + 9\%e^{2t} C4 + 27\%e^{2t} C3 + 3\%e^{2t} C2 + 10\%e^{2t} C1 \right) \sin(3t) + \left(\left(3\%e^{2t} C4 + 9\%e^{2t} C3 + \%e^{2t} C2 \right) \sin(3t) + \left(3\%e^{2t} C4 + 9\%e^{2t} C3 + \%e^{2t} C2 \right) \cos(3t) \right) \end{bmatrix} \quad (\text{sol})$$

(% i13) sol[1,1];

$$\left(\left(- \left(3e^{2t}C4 \right) - 9e^{2t}C3 - e^{2t}C2 - 3e^{2t}C1 \right) t - e^{2t}C4 - 3e^{2t}C3 \right) \sin(3t) + \left(e^{2t}C3t + e^{2t}C1 \right) \cos(3t) \quad (\% \text{ o13})$$

(% i14) sol[2,1];

$$\left(\left(9e^{2t}C4 + 24e^{2t}C3 + 3e^{2t}C2 + 9e^{2t}C1 \right) t + 3e^{2t}C4 + 10e^{2t}C3 - 3e^{2t}C1 \right) \sin(3t) + \left(\left(- \left(9e^{2t}C4 \right) - \right) \right) \cos(3t) \quad (\% \text{ o14})$$

(% i15) sol[3,1];

$$\left(- \left(3e^{2t}C4 \right) - 9e^{2t}C3 - e^{2t}C2 - 3e^{2t}C1 \right) \sin(3t) + e^{2t}C3 \cos(3t) \quad (\% \text{ o15})$$

(% i16) sol[4,1];

$$\left(e^{2t}C3t + 9e^{2t}C4 + 27e^{2t}C3 + 3e^{2t}C2 + 10e^{2t}C1 \right) \sin(3t) + \left(\left(3e^{2t}C4 + 9e^{2t}C3 + e^{2t}C2 + 3e^{2t}C1 \right) t \right) \cos(3t) \quad (\% \text{ o16})$$

Test

(% i18) diff(sol,t)-A.sol\$
expand(%);

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\% \text{ o18})$$

21 odeM_ic

Function: odeM_ic(A, F, t, t0, Y0) Find solutions of initial problem for linear system of ODEs in matrix form :
 $Y' = AY + F$, $Y(t_0) = Y_0$.

21.1 Example 1 : $Y' = AY + F$, $Y(0)=Y_0$

(% i19) A:matrix([2,-4],[2,-2]);

$$\begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \quad (\text{A})$$

(% i20) F:transpose([4*e^(-2*t),0]);

$$\begin{bmatrix} 4e^{-(2t)} \\ 0 \end{bmatrix} \quad (\text{F})$$

(% i21) Y0:transpose([0,0]);

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{Y0})$$

(% i22) sol:odeM_ic(A,F,t,0,Y0);

$$\begin{bmatrix} 2 \sin(2t) \\ \sin(2t) - \cos(2t) + e^{-(2t)} \end{bmatrix} \quad (\text{sol})$$

Test

(% i24) diff(sol,t)-A.sol-F\$ expand(%);

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\% \text{ o24})$$

(% i25) ev(sol,t=0);

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\% \text{ o25})$$

21.2 Example 2 : $Y' = AY$, $Y(0)=\text{transpose}([15,35,55,75])$.

(% i26) A:matrix([4,1,1,7],[1,4,10,1],[1,10,4,1],[7,1,1,4]);

$$\begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} \quad (\text{A})$$

(% i27) F:transpose([0,0,0,0])\$

(% i28) Y0:transpose([15,35,55,75]);

$$\begin{bmatrix} 15 \\ 35 \\ 55 \\ 75 \end{bmatrix} \quad (\text{Y0})$$

(% i29) charpoly(A, x),factor;

$$(x - 15) (x - 10) (x + 3) (x + 6) \quad (\% \text{ o29})$$

(% i30) sol:odeM_ic(A,F,t,0,Y0);

$$\begin{bmatrix} 27\%e^{15t} + 18\%e^{10t} - 30\%e^{-(3t)} \\ 54\%e^{15t} - 9\%e^{10t} - 10\%e^{-(6t)} \\ 54\%e^{15t} - 9\%e^{10t} + 10\%e^{-(6t)} \\ 27\%e^{15t} + 18\%e^{10t} + 30\%e^{-(3t)} \end{bmatrix} \quad (\text{sol})$$

Test

(% i32) diff(sol,t)-A.sol\$
expand(%);

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\% \text{ o32})$$

(% i33) ev(sol,t=0);

$$\begin{bmatrix} 15 \\ 35 \\ 55 \\ 75 \end{bmatrix} \quad (\% \text{ o33})$$

22 matrix_exp

Function: matrix_exp(A,t)Returns matrix exponential $e^{(A \star t)}$ computed via Laplace transforms.

22.1 Example 1

(% i34) A:matrix([1,1],[0,1]);

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (\text{A})$$

(% i35) matrix_exp(A,t);

$$\begin{bmatrix} \%e^t & \%e^t t \\ 0 & \%e^t \end{bmatrix} \quad (\% \text{ o35})$$

(% i36) e^`A=matrix_exp(A,1);

$$e^A = \begin{bmatrix} \%e & \%e \\ 0 & \%e \end{bmatrix} \quad (\% \text{ o36})$$

22.2 Example 2

(% i37) A:matrix([21,17,6],[-5,-1,-6],[4,4,16]);

$$\begin{bmatrix} 21 & 17 & 6 \\ -5 & -1 & -6 \\ 4 & 4 & 16 \end{bmatrix} \quad (\text{A})$$

(% i38) e^`A=matrix_exp(A,1);

$$e^A = \begin{bmatrix} \frac{13\%e^{16}}{4} - \frac{\%e^4}{4} & \frac{13\%e^{16}}{4} - \frac{5\%e^4}{4} & \frac{\%e^{16}}{2} - \frac{\%e^4}{2} \\ \frac{\%e^4}{4} - \frac{9\%e^{16}}{4} & \frac{5\%e^4}{4} - \frac{9\%e^{16}}{4} & \frac{\%e^4}{2} - \frac{\%e^{16}}{2} \\ 4\%e^{16} & 4\%e^{16} & \%e^{16} \end{bmatrix} \quad (\% \text{ o38})$$

23 odelinsys

Function: odelinsys(A, F, x, x0, Y0) Find solutions of initial problem for linear system of ODEs in matrix form:
 $Y' = AY + F$, $Y(x_0) = Y_0$.

(% i39) load(diag)\$

23.1 Example 1 : Solve $Y' = AY + F$, $Y(0) = Y_0$

(% i40) A:matrix([1,3],[-1,5]);

$$\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \quad (\text{A})$$

(% i42) F:transpose([-x,2*x])\$ Y0:transpose([3,1])\$

(% i43) sol:odelinsys(A,F,x,0,Y0);

$$\begin{bmatrix} \frac{11x}{8} + \frac{7\%e^{4x}}{32} + \frac{15\%e^{2x}}{8} + \frac{29}{32} \\ -\left(\frac{x}{8}\right) + \frac{7\%e^{4x}}{32} + \frac{5\%e^{2x}}{8} + \frac{5}{32} \end{bmatrix} \quad (\text{sol})$$

Test

(% i44) diff(sol,x)-A.sol-F,expand;

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\% \text{ o44})$$

23.2 Solve $Y' = AY$

(% i45) `A:matrix([4,-1,0],[3,1,-1],[1,0,1]);`

$$\begin{bmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad (\text{A})$$

(% i46) `sol:odelinsys(A,[0,0,0],t,0,[C1,C2,C3]),factor;`

$$\begin{bmatrix} \frac{{\%e}^{2t}(C3t^2 - C2t^2 + C1t^2 - 2C2t + 4C1t + 2C1)}{2} \\ {\%e}^{2t} \left(C3t^2 - C2t^2 + C1t^2 - C3t - C2t + 3C1t + C2 \right) \\ \frac{{\%e}^{2t}(C3t^2 - C2t^2 + C1t^2 - 2C3t + 2C1t + 2C3)}{2} \end{bmatrix} \quad (\text{sol})$$

24 wronskian

Function: `wronskian([f_1, ..., f_n], x)` Returns the Wronskian matrix of the list of expressions $[f_1, \dots, f_n]$ in the variable x .

(% i47) `kill(all)$`

(% i1) `load(odes2)$`

24.1 Example 1

(% i2) `wronskian([f(x),g(x),h(x)],x);`

$$\begin{bmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx} f(x) & \frac{d}{dx} g(x) & \frac{d}{dx} h(x) \\ \frac{d^2}{dx^2} f(x) & \frac{d^2}{dx^2} g(x) & \frac{d^2}{dx^2} h(x) \end{bmatrix} \quad (\% \text{ o2})$$

24.2 Example 2

Form a linear homogeneous differential equation, knowing its fundamental system of solutions : $y_1 = x$, $y_2 = x^3$.

(% i3) `depends(y,x);`

$$[y(x)] \quad (\% \text{ o3})$$

(% i4) `wronskian([x,x^3,y],x);`

$$\begin{bmatrix} x & x^3 & y \\ 1 & 3x^2 & \frac{d}{dx} y \\ 0 & 6x & \frac{d^2}{dx^2} y \end{bmatrix} \quad (\% \text{ o4})$$

(% i5) `determinant(%)=0;`

$$x \left(3x^2 \left(\frac{d^2}{dx^2} y \right) - 6x \left(\frac{d}{dx} y \right) \right) - x^3 \left(\frac{d^2}{dx^2} y \right) + 6xy = 0 \quad (\% \text{ o5})$$

(% i6) `eq:expand(%/x/2);`

$$x^2 \left(\frac{d^2}{dx^2} y \right) - 3x \left(\frac{d}{dx} y \right) + 3y = 0 \quad (\text{eq})$$

(% i7) `ode2(eq,y,x);`

$$y = \%k1x^3 + \%k2x \quad (\% \text{ o7})$$